Characterization of $\mathbf{C}^n$ by its Automorphism Group*†

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We show that if the group of holomorphic automorphisms of a connected Stein manifold M is isomorphic to that of $\mathbb{C}^n$ as a topological group equipped with the compact-open topology, then M is biholomorphically equivalent to $\mathbb{C}^n$.

1 Introduction

Let M be a connected complex manifold of dimension n and let $\mathrm{Aut}(M)$ denote the group of holomorphic automorphisms of M. The group $\mathrm{Aut}(M)$ is a topological group equipped with the natural compact-open topology. We are interested in the problem of characterizing M by $\mathrm{Aut}(M)$. This problem becomes particularly intriguing when $\mathrm{Aut}(M)$ is infinite-dimensional.

Let, for example, $M = \mathbb{C}^n$ and suppose that M' is such that $\mathrm{Aut}(M')$ is isomorphic as a topological group to $\mathrm{Aut}(\mathbb{C}^n)$; is it then true that M' is biholomorphically equivalent to $\mathbb{C}^n$?

In [IK] we gave a positive answer to the above question, and the proof there followed from a general classification of all connected n-dimensional complex manifolds that admit effective actions of the unitary group U_n by holomorphic transformations. In this paper we give a simpler proof in the case of Stein manifolds. This proof does not require considering the whole group U_n, but relies only on linearization of the induced action of the torus $\mathbb{T}^n \subset U_n$ [BBD].

In this paper we prove the following theorem.

THEOREM 1.1 *Let M be a connected Stein manifold of dimension n and suppose that $Aut(M)$ and $Aut(\mathbb{C}^n)$ are isomorphic as topological groups (both groups are considered with the compact-open topology). Then M is biholomorphically equivalent to $\mathbb{C}^n$.*

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2 Proof of Theorem

The theorem is obvious for $n = 1$, so we assume that $n \geq 2$. Let $\Phi : \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(M)$ be an isomorphism. The group $\mathrm{Aut}(\mathbb{C}^n)$ contains the subgroup $\mathbb{C}^{*n}$, i.e., all transformations of the form

$$(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n), \tag{2.1}$$

with $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^{*n}$ and $z := (z_1, \dots, z_n) \in \mathbb{C}^n$. Therefore $\mathbb{C}^{*n}$ acts on M with the action mapping $F : \mathbb{C}^{*n} \times M \to M$ defined as follows:

$$F(\lambda, p) := \Phi(\lambda)(p),$$

for $\lambda \in \mathbb{C}^{*n}$, $p \in M$. This action is clearly effective on M. Since F is continuous in (λ, p) and holomorphic in p, it is in fact real-analytic in (λ, p) [BM].

We will now prove the following proposition which is of independent interest and holds for manifolds more general than Stein manifolds.

Proposition 2.1 *Let M be a connected manifold of complex dimension n whose envelope of holomorphy is smooth, and suppose that M admits an effective action of $\mathbb{C}^{*n}$ by holomorphic transformations. Then M is biholomorphically equivalent to either $\mathbb{C}^n$, or $\mathbb{C}^n \setminus \{0\}$, or $\mathbb{C}^n$ without some coordinate hyperplanes:*

$$\mathbb{C}^n \setminus \cup_{k=1}^r \{z_{i_k} = 0\}, \qquad r \geq 1. \tag{2.2}$$

Proof: Let as above $F : \mathbb{C}^{*n} \times M \to M$ denote the action of $\mathbb{C}^{*n}$ on M. Consider the restriction of the action to the torus $\mathbb{T}^n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^{*n} : |\lambda_j| = 1 \text{ for all } j\}$. It follows from [BBD] that there exists a holomorphic embedding $\alpha : M \to \mathbb{C}^n$ such that $D := \alpha(M)$ is a Reinhardt domain and the induced action $G(\lambda, z) = (G_1(\lambda, z), \dots, G_n(\lambda, z)) := \alpha(F(\lambda, \alpha^{-1}(z)))$ of $\mathbb{T}^n$ on D has the form

$$G_j(\lambda, z) = e^{i(a_{j1}\lambda_1 + \ldots + a_{jn}\lambda_n)} z_j,$$

where $\lambda \in \mathbb{T}^n$, $z \in D$ and a_{jk} are fixed integers such that $\det(a_{jk}) = \pm 1$.

Let $F_\alpha : \mathbb{C}^{*n} \times D \to D$ be the induced action of $\mathbb{C}^{*n}$ on D:

$$F_\alpha(\lambda, z) := \alpha(F(\lambda, \alpha^{-1}(z))),$$

for $\lambda \in \mathbb{C}^{*n}$, $z \in D$. Denote by $\Phi_\alpha : \mathbb{C}^{*n} \to \mathrm{Aut}(D)$ the corresponding homomorphism:

$$\Phi_\alpha(\lambda)(z) = F_\alpha(\lambda, z),$$

for $\lambda \in \mathbb{C}^{*n}$, $z \in D$. Since G and F_α coincide on $\mathbb{T}^n$, it follows that $\Phi_\alpha(\mathbb{T}^n)$ consists precisely of all mappings of the form (2.1) with $|\lambda_1| = \ldots = |\lambda_n| = 1$, i.e., $\Phi_\alpha(\mathbb{T}^n) = \mathbb{T}^n$. Let $C(\mathbb{T}^n)$ denote the centralizer of $\mathbb{T}^n$ in $\mathrm{Aut}(D)$, i.e.,

$$C(\mathbb{T}^n) := \{f \in \mathrm{Aut}(D) : f \circ t = t \circ f \quad \text{for all } t \in \mathbb{T}^n\}.$$

Since $\mathbb{C}^{*n}$ is commutative, we have $\Phi_\alpha(\mathbb{C}^{*n}) \subset C(\mathbb{T}^n)$.

We now need the following lemma.

Lemma 2.2 *Every element of $C(\mathbb{T}^n)$ has the the form (2.1), i.e., $C(\mathbb{T}^n) \subset \mathbb{C}^{*n}$.*

Proof: Let $f = (f_1, \ldots, f_n) \in C(\mathbb{T}^n)$. Then we have

$$f_j(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) = e^{i\theta_j} f_j(z_1, \ldots, z_n), \qquad j = 1, \ldots, n, \tag{2.3}$$

for all $\theta_1, \ldots, \theta_n \in \mathbb{R}$ and $z \in D$. In particular, for every fixed j we have

$$f_j(z_1, \ldots, z_{k-1}, e^{i\theta} z_k, z_{k+1}, \ldots, z_n) = f_j(z_1, \ldots, z_n),$$

for all $\theta \in \mathbb{R}$, $k \neq j$, which implies that f_j depends only on z_j (we will write $f_j = f_j(z_j)$). Then (2.3) gives

$$f_j(e^{i\theta} z_j) = e^{i\theta} f_j(z_j), \tag{2.4}$$

for all $\theta \in \mathbb{R}$. Differentiating (2.4) with respect to z_j we get

$$f_j'(e^{i\theta} z_j) = f_j'(z_j),$$

which implies that $f_j' \equiv \text{const}$ and thus $f_j = \lambda_j z_j$, $\lambda_j \in \mathbb{C}^*$.

The lemma is proved. $\square$

Lemma 2.2 gives that Φ_α is a continuous homomorphism from $\mathbb{C}^{*n}$ into itself, and thus is a Lie group homomorphism (see, e.g, [W]). Further, since Φ_α is injective, it is in fact an automorphism of $\mathbb{C}^{*n}$. In particular, D is

invariant under all mappings of the form (2.1), and thus D is either $\mathbb{C}^n$ or, $\mathbb{C}^n \setminus \{0\}$, or a domain of the form (2.2).

The proposition is proved. $\square$

We will now show that the automorphism groups of $\mathbb{C}^n$ and any domain of the form (2.2) are not isomorphic as topological groups. This is a consequence of the following observation.

Lemma 2.3 *For $n \geq 2$ we have*

(i) $Aut(\mathbb{C}^n)$ is connected;

(ii) If D is a domain of the form (2.2), then $Aut(D)$ is disconnected.

Proof: Following [AL], we consider special automorphisms of $\mathbb{C}^n$ called overshears:

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, f(z_1, \ldots, z_{n-1}) + h(z_1, \ldots, z_{n-1})z_n), \qquad (2.5)$$

where f, h are entire functions on $\mathbb{C}^{n-1}$ and h is nowhere zero. Let $G(\mathbb{C}^n)$ denote the subgroup of $\mathrm{Aut}(\mathbb{C}^n)$ generated by overshears (2.5) and permutations of coordinates. It is proved in [AL] that $G(\mathbb{C}^n)$ is dense in $\mathrm{Aut}(\mathbb{C}^n)$. We will show that every element of $G(\mathbb{C}^n)$ can be joined with the identity by a continuous path in $\mathrm{Aut}(\mathbb{C}^n)$. For a mapping of the form (2.5) we choose a path $\gamma(t)$ as follows:

$$\gamma(t)(z_1, \ldots, z_n) := (z_1, \ldots, z_{n-1}, (1-t)f(z_1, \ldots, z_{n-1}) + h^{1-t}(z_1, \ldots, z_{n-1})z_n),$$
$$0 \leq t \leq 1.$$

Further, for the permutation of z_j and z_k we choose (assuming $j < k$)

$$\begin{aligned} &\gamma(t)(z_1, \ldots, z_n) := \\ &(z_1, \ldots, z_{j-1}, (1-t)z_k + tz_j, z_{j+1}, \ldots, z_{k-1}, \\ &tz_k + [(1-t) + if(t)]z_j, z_{k+1}, \ldots, z_n), \qquad 0 \leq t \leq 1, \end{aligned}$$

where f is a real-valued continuous function on $[0,1]$ such that $f(0) = f(1) = 0$ and $f(1/2) \neq 0$.

Therefore, $G(\mathbb{C}^n)$ lies in the identity component of $\mathrm{Aut}(\mathbb{C}^n)$ and hence $\mathrm{Aut}(\mathbb{C}^n)$ does not in fact have any other connected components. Thus, $\mathrm{Aut}(\mathbb{C}^n)$ is connected, and (i) is proved.

Let D be a domain of the form (2.2). Choose $1 \leq s \leq r$ and $p \in D$. Let L_p be the complex affine line in $\mathbb{C}^n$ orthogonal to the hyperplane $\{z_s = 0\}$ and passing through p. Denote by q the point of intersection of L_p and $\{z_s = 0\}$. Next, we choose a closed curve Γ in $L_p \cap D$ around q and define a subset of $\mathrm{Aut}(D)$ as:

$$A := \left\{ (f_1, \ldots, f_n) \in \mathrm{Aut}(D) : \frac{1}{2\pi i} \int_\Gamma \frac{\hat{f}'_s}{\hat{f}_s} \, dz_s < 0 \right\},$$

where $\hat{f}_s := f_s|_{L_p}$ and $\hat{f}'_s$ denotes the derivative of $\hat{f}_s$ with respect to z_s. The subset A is clearly open in $\mathrm{Aut}(D)$. It is also closed in $\mathrm{Aut}(D)$ since the integrals in its definition are integers. It is non-empty since it contains the automorphism

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{s-1}, \frac{1}{z_s}, z_{s+1}, \ldots, z_n).$$

And obviously $A \neq \mathrm{Aut}(D)$ since the identity mapping is not in A. Therefore, $\mathrm{Aut}(D)$ is disconnected, and (ii) is proved. □

It now follows from Lemma 2.3 that $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Aut}(D)$ are not isomorphic as topological groups, and hence M is not equivalent to D. Further, since M is Stein and $\mathbb{C}^n \setminus \{0\}$ is not, Proposition 2.1 gives that M is biholomorphically equivalent to $\mathbb{C}^n$, and the theorem is proved. □

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